

Quantum temporal probabilities in tunneling systems: I. Tunneling times in quantum field theory

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Abstract

We study the temporal aspects of quantum tunneling as manifested in time-of-arrival experiments, in which the detected particle tunnels through a potential barrier. In particular, we present a general method for constructing temporal probabilities in tunneling systems that (i) defines ‘classical’ time observables for quantum systems and (ii) applies to relativistic particles interacting through quantum fields. We show that the relevant probabilities are defined in terms of specific correlations functions of the quantum field associated to tunneling particles. We construct a probability distribution with respect to the time of particle detection that contains all information about the temporal aspects of the tunneling process. In specific cases, this probability distribution leads to the definition of a delay time that, for parity-symmetric potentials, reduces to the phase time of Bohm and Wigner. We apply our results to piecewise constant potentials, by deriving the appropriate junction conditions on the points of discontinuity. For the double square potential, in particular, we demonstrate the existence of (at least) two physically relevant time parameters, the delay time and a decay rate that describes the escape of particles trapped in the inter-barrier region.

1 Introduction

The issue of the time that a quantum particle takes to tunnel through a potential barrier has been studied since the early days of quantum mechanics [1, 2]. The search for an answer to this question has led to several different candidates for the tunneling time—for reviews, see Ref.[3]—rather than to a single expression derived unambiguously from first principles. Many existing definitions imply that tunneling times saturate in the opaque-barrier limit, thus suggesting superluminal speeds for particles traversing sufficiently long barriers, a phenomenon known as the Hartmann effect [4].

In this paper, we identify tunneling time through its association with quantum *temporal* observables, i.e., time variables whose value can be determined in specific experiments. A quantum temporal observable is a random variable, so its determination requires the construction of a probability density rather than the specification of a single number. Hence, any temporal parameters that characterize the tunneling process must be identified in terms of a probability distribution associated to physical measurements on the tunneling particles. In particular, we propose a definition of tunneling time in terms of the time instant that a detector, located far from the barrier, records tunneling particles, i.e., that a definition in terms of time-of-arrival measurements. In our

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opinion, this is the most natural operational definition of tunneling time, because a remote detector does not interfere with the tunneling process—unlike existing proposed definitions of tunneling time in terms of external actions on the barrier region [5, 6].

A time-of-arrival experiment typically involves a particle source and a particle detector separated by distance L at rest with respect to another. Source and detector are equipped with a pair of synchronized clocks. The time of arrival is the difference t between the clock readings of detection and emission respectively. This ‘classical’ definition of the time of arrival can be extended to quantum particles using the Quantum Temporal Probabilities (QTP) method [13]. The main difference is that for quantum particles, the time difference t is a random variable that may take different values in different runs of the experiment. Thus, a quantum description involves the construction of a probability density $P(L, t)$ for the time of arrival. When a potential barrier is placed along the line between emitter and detector, all information about the temporal aspects of tunneling (the tunneling time and any other physically relevant parameter) is contained in the probability density $P(L, t)$ [7]—for other approaches to tunneling time in terms of time-of-arrival measurements, see Refs. [8, 9, 10].

The construction of time-of-arrival probabilities associated to tunneling systems is a non-trivial task. In fact, the definition of time-of-arrival probabilities has run into ambiguities analogous to the ones in the definition of tunneling time—for reviews, see Refs. [11, 12]. The difficulty originates from the special role of time in quantum mechanics, where the time t appearing in Schrödinger’s equation is an external parameter and not as an ordinary observable, like position or momentum. As a result, the squared modulus of the time-evolved wave-function, $|\psi(x, t)|^2$, is not a density with respect to t , and, hence, it cannot serve as a definition for the required probabilities.

In this work, we employ a general method for the construction of quantum temporal probabilities associated to any experimental configuration, that was developed in Ref. [13]. The QTP method incorporates the detector degrees of freedom into the quantum description, so that the temporal probabilities are always defined with respect to specific experimental set-ups. The key idea is to distinguish between the roles of time as a parameter to Schrödinger’s equation and as a label of the causal ordering of events [14, 15]. This important distinction leads to the definition of quantum temporal observables. In particular, we identify the time of a detection event as a coarse-grained quasi-classical variable [16, 17] associated to macroscopic records of observation. The time variables in QTP correspond to macroscopic observable magnitudes, such as the coincidence of a detector ‘click’ with the reading of a clock external to the system.

A key property of the QTP method is that it applies to *any quantum system*, including relativistic quantum fields. Besides time-of-arrival probabilities, the method has been applied to the modeling of particle detectors in high-energy processes [13], and to the study of temporal correlations in accelerated detectors [18]. Earlier versions of QTP [19] have been employed in studies of non-relativistic tunneling times [7], non-exponential decays [20] and time-extended measurements [21].

Using the QTP method, we derive a probability density $P(L, t)$ describing time-of-arrival measurements on relativistic particles that tunnel through a potential barrier. The interactions of the particles with the barrier and the detector are described in terms of Quantum Field Theory (QFT). The definition of relativistic tunneling times in terms of QFT (rather than relativistic quantum mechanics [22]) is important, because a QFT treatment of tunneling times is essential for a resolution of the superluminality paradox [23]. The reason is that superluminal speeds imply a violation of the principle of local causality, and this principle is implemented in quantum theory only if the interactions are expressed in terms of local quantum fields [24, 25].

The probability density $P(L, t)$ derived here incorporates all temporal aspects of the tunneling process. For a specific class of potentials and initial states, the only timescale relevant to tunneling is a temporal variable t_d . This variable describes the delay in the transit time of particles crossing

through the barrier in comparison to the transit time of the same distance in absence of the barrier. For parity-symmetric potential, t_d coincides with the Bohm-Wigner phase time [26], obtained from the asymptotic analysis of wave-packets. However, our identification of the delay time does not employ non quantum-invariant notions, such as the peak of the wave packet or its center of mass, but follows from the structure of the probability density $P(L, t)$, in specific regimes [27].

In general, a single time parameter, such as t_d , does not capture all temporal aspects of the tunneling process. We demonstrate this point by studying tunneling through a double square barrier. We find that the probability distribution $P(L, t)$ is characterized by two distinct timescales, a delay time t_d , as described earlier, but also a decay time Γ^{-1} that determines the escape rate of particles trapped between the two barriers. Previous work on this system, led by the *a priori* assumption of a single-time scale governing tunneling, had led to a very different physical description of the tunneling process, according to which the tunneling time is largely independent of the inter-barrier distance [28, 29].

The main results of the paper are the following.

1. We present a general methodology for defining the time-of-arrival probabilities $P(L, t)$ in tunneling set-ups that is valid for any potential barrier and for any method of particle detection. We derive the probability distribution $P(L, t)$ for relativistic particles interacting through quantum fields. We show that $P(L, t)$ is obtained from a suitable two-point function of the quantum field smeared with a kernel that depends on the physics of the detection process.
2. For the standard case of a potential barrier model by a background classical field, we derive an simple expression for the probability distribution $P(L, t)$ as a positive linear functional of the initial state. The probability distribution $P(L, t)$ can be explicitly constructed from the knowledge of the transmission and reflection amplitudes of the potential.
3. We find that the total transmission probability coincides with the square modulus of the transmission amplitude associated to the potential only if the potential is parity symmetric. In general, they differ. This is because the QFT propagator through which the probability distribution $P(L, t)$ is defined involves contributions from both left-moving and right-moving eigenstates of the single-particle Hamiltonian.
4. We construct the probability distribution $P(L, t)$ explicitly for piecewise constant potentials, by deriving the appropriate junction conditions on the points of discontinuity of the potential. For the double square potential, in particular, we demonstrate explicitly that a single time parameter does not suffice to capture all information about the temporal aspects of tunneling.

In a companion paper, we analyze in detail the superluminality paradox in tunneling, in light of the present results [23].

The structure of the paper is the following. In Sec. 2, we formulate the quantum tunneling of particles through a barrier in a quantum field theory language. In Sec. 3, we derive a general expression for the probability density $P(L, t)$ of detection of particles through a potential barrier. In Sec. 4, we construct explicitly the probability density $P(L, t)$ for piecewise-constant potentials. In Sec. 5, we summarize our results and discuss their further applications.

2 QFT formulation of tunneling

In this section, we formulate tunneling of particles through a potential barrier in a quantum field theory language. In particular, we consider a complex scalar field $\phi(x)$, corresponding to particles

of mass m and charge e , interacting with a background static electric field. The electromagnetic potential is $A_\mu(x) = (A_0(x), 0)$, where $A_0(x)$ differs from zero only in a spatial region D . For simplicity, we restrict our considerations to one spatial dimension, and we choose the origin of the coordinate system so that $D = [-d/2, d/2]$, where d is the length of the barrier.

The classical Lagrangian density for the scalar field $\phi(x)$ is

$$\mathcal{L}(\phi, \phi^*) = |\dot{\phi} - iV\phi|^2 - |\partial_x \phi|^2 - m^2|\phi|^2, \quad (1)$$

where $V(x) = eA_0(x)$.

The associated Hamiltonian density \mathcal{H} is

$$\mathcal{H} = |\pi|^2 + |\partial_x \phi|^2 + m^2|\phi|^2 + iV(\pi\phi - \pi^*\phi^*), \quad (2)$$

where π and π^* are conjugate field variables to ϕ and ϕ^* respectively.

In order to quantize the field, we first introduce two copies H_1 and \bar{H}_1 of the Hilbert space $\mathcal{L}^2(R, dx)$ of square integrable functions on the real line. H_1 and \bar{H}_1 are associated to a single particle and to a single antiparticle, respectively. The Hilbert space of the field is defined as $\mathcal{F}(H_1) \otimes \mathcal{F}(\bar{H}_1)$. $\mathcal{F}(H)$ stands for the bosonic Fock space associated to the Hilbert space H ,

$$\mathcal{F}(H) = C \oplus H \oplus (H \otimes H)_S \oplus (H \otimes H \otimes H)_S \oplus \dots, \quad (3)$$

where S denotes symmetrization.

The creation and annihilation operators $\hat{a}(x)$ and $\hat{a}^\dagger(x)$ for particles and $\hat{b}(x)$ and $\hat{b}^\dagger(x)$ for antiparticles are standardly defined on the Fock space $\mathcal{F}(H_1) \otimes \mathcal{F}(\bar{H}_1)$. They satisfy the commutation relations

$$[\hat{a}(x), \hat{a}^\dagger(x')] = \delta(x - x') \quad [\hat{b}(x), \hat{b}^\dagger(x')] = \delta(x - x'), \quad (4)$$

with all other commutators vanishing.

The creation and annihilation operators are also expressed in their smeared form

$$\hat{a}(f) = \int dx \hat{a}(x) f(x), \quad \hat{a}^\dagger(f) = \int dx \hat{a}^\dagger(x) f(x) \quad (5)$$

$$\hat{b}(f) = \int dx \hat{b}(x) f(x), \quad \hat{b}^\dagger(f) = \int dx \hat{b}^\dagger(x) f(x), \quad (6)$$

where f is a square-integrable function on R .

In order to represent the Hamiltonian Eq. (2) as an operator on the Fock space $\mathcal{F}(H_1) \otimes \mathcal{F}(\bar{H}_1)$ we introduce the field operators $\hat{\phi}(x), \hat{\phi}^\dagger(x)$ and their conjugate momenta $\hat{\pi}(x), \hat{\pi}^\dagger(x)$. The field operators are defined in their smeared form as

$$\hat{\phi}(f) := \frac{1}{\sqrt{2}} \left(\hat{a}(h_0^{-1/2} f) + \hat{b}^\dagger(h_0^{-1/2} f) \right) \quad \hat{\pi}(f) := \frac{1}{\sqrt{2}i} \left(\hat{a}(h_0^{1/2} f) - \hat{a}^\dagger(h_0^{1/2} f) \right) \quad (7)$$

$$\hat{\phi}^\dagger(f) := \frac{1}{\sqrt{2}} \left(\hat{b}(h_0^{-1/2} f) + \hat{a}^\dagger(h_0^{-1/2} f) \right) \quad \hat{\pi}^\dagger(f) := \frac{1}{\sqrt{2}i} \left(\hat{b}(h_0^{1/2} f) - \hat{b}^\dagger(h_0^{1/2} f) \right). \quad (8)$$

where $h_0 = \sqrt{-\partial_x^2 + m^2}$ is the Hamiltonian operator for a free relativistic particle, defined on the Hilbert space $H_1 = \bar{H}_1$.

We construct the Hamiltonian operator \hat{H} for the field by substituting the field operators above into the classical Hamiltonian density (2). After normal ordering,

$$\hat{H} = \int dx dx' \left(\hat{a}^\dagger(x) h_1(x, x') \hat{a}(x) + \hat{b}^\dagger(x) h_2(x, x') \hat{b}(x) \right). \quad (9)$$

In Eq. (9), $h_1(x, x')$ are matrix elements in the position basis of a Hamiltonian operator h_1 on the Hilbert space H_1 of a single particle and $h_2(x, x')$ are matrix elements in the position basis of a Hamiltonian operator h_2 on the Hilbert space \bar{H}_1 of a single antiparticle. The operators $h_{1,2}$ are defined as

$$h_1 = h_0 + \tilde{V}, \quad h_2 = h_0 - \tilde{V} \quad (10)$$

where

$$\tilde{V} := \frac{1}{2} \left(h_0^{1/2} V h_0^{-1/2} + h_0^{1/2} V h_0^{-1/2} \right) = V + [[V, h_0^{1/2}], h_0^{-1/2}], \quad (11)$$

is the potential operator, that includes a QFT correction term. The sign difference between h_1 and h_2 indicates the fact that a potential barrier for particles is a potential well for anti-particles.

In what follows, we will consider a positive-valued potential $V(x)$, such that the corresponding single-particle Hamiltonian h_1 has purely continuous spectrum. The eigenstates of h_1 are characterized by energies $E > m$ and they have a double degeneracy. We denote the pair of eigenstates corresponding to the same value E of energy as f_{k+} and f_{k-} , where $k = \sqrt{E^2 - m^2} > 0$. The eigenstates $f_{k+}(x)$ correspond to only positive momentum flux at $x = \infty$ and they are of the form

$$f_{k+}(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} e^{ikx} + R_k e^{-ikx} & x < -\frac{d}{2} \\ T_k e^{ikx} & x > \frac{d}{2} \end{cases} \quad (12)$$

In Eq. (12), T_k and R_k are the usual transmission and reflection coefficients for a right-moving plane wave of momentum k . The eigenstates f_{k-} are orthogonal to f_{k+} and they satisfy

$$f_{k-}(x) = \frac{1}{\sqrt{1 - w_k^2}} [-w_k f_{k+}(x) + f_{k+}(-x)], \quad (13)$$

where $w_k = \frac{1}{2}(T_k^* R_k + T_k R_k^*)$ is essentially the overlap between f_{k+} and its parity-transform. For parity-symmetric potentials, i.e., for potentials such that $V(x) = V(-x)$, the coefficient w_k vanishes and $f_{k-}(x) = f_{k+}(-x)$.

The Hamiltonian h_2 , describing anti-particles, has both continuous ($E > m$) and discrete spectrum ($E < m$). We denote the continuous-spectrum eigenstates as g_{k+} and g_{k-} . They are similar in structure to the generalized eigenstates $f_{k\pm}$ of h_1 . Their form is not relevant to the purposes of this paper.

We denote the discrete-spectrum eigenstates of h_2 as $g_n(x)$, where the label n runs into a finite set of values. The eigenstates $g_n(x)$ decay exponentially outside the barrier region.

$$g_n(x) = \begin{cases} A_n e^{\gamma_n x} & x < -d/2 \\ B_n e^{-\gamma_n x} & x > d/2 \end{cases}, \quad (14)$$

for some positive constants A_n and B_n , and

$$\gamma_n = \sqrt{m^2 - E_n^2}. \quad (15)$$

For parity-symmetric potentials, $B_n = P(n)A_n$, where $P(n)$ is the parity of the eigenstate $g_n(x)$.

Next, we define the energy-basis creation and annihilation operators,

$$\hat{a}_{k\pm} = \int dx \hat{a}(x) f_{k\pm}^*(x) \quad \hat{b}_{k\pm} = \int dx \hat{b}(x) g_{k\pm}^*(x) \quad \hat{b}_n = \int dx \hat{b}(x) g_n^*(x) \quad (16)$$

$$\hat{a}_{k\pm}^\dagger = \int dx \hat{a}^\dagger(x) f_{k\pm}(x) \quad \hat{b}_{k\pm}^\dagger = \int dx \hat{b}^\dagger(x) g_{k\pm}(x) \quad \hat{b}_n^\dagger = \int dx \hat{b}^\dagger(x) g_n(x). \quad (17)$$

Then, the Heisenberg-picture field operators are

$$\hat{\phi}(x, t) = \sum_{\sigma=-}^+ \int_0^\infty \frac{dk}{\sqrt{2E_k}} \left(\hat{a}_{k,\sigma} f_{k,\sigma}(x) e^{-iE_k t} + \hat{b}_{k,\sigma}^* g_{k,\sigma}^*(x) e^{iE_k t} \right) + \sum_n \frac{1}{\sqrt{2E_n}} \hat{b}_n^* g_n^*(x) e^{iE_n t} \quad (18)$$

$$\hat{\phi}^\dagger(x, t) = \sum_{\sigma=-}^+ \int_0^\infty \frac{dk}{\sqrt{2E_k}} \left(\hat{b}_{k,\sigma} g_{k,\sigma}(x) e^{-iE_k t} + \hat{a}_{k,\sigma}^\dagger f_{k,\sigma}^*(x) e^{iE_k t} \right) + \sum_n \frac{1}{\sqrt{2E_n}} \hat{b}_n g_n(x) e^{-iE_n t}. \quad (19)$$

Eqs. (18) and (19) are the basis for the QFT treatment of tunneling time in the following section.

3 Probabilities for tunneling time

In this section, we derive a general expression for the detection probability $P(L, t)$ for relativistic particles tunneling through a potential barrier. First, we present a brief review of the QTP method, and then we derive an equation that relates the detection probability to the correlation functions of a quantum field theory. Finally, we specialize to the scalar field system described in Sec. 2, and we express the probability $P(L, t)$ in terms of the transmission and reflection coefficient associated to the potential barrier.

3.1 Background

Our approach is based on the general methodology for constructing time-of-arrival probabilities associated to general detectors, that was developed in Ref. [13] (the Quantum Temporal Probabilities method). First, we give a brief review of the QTP method, setting up the background for its application to the tunneling-time problem in Secs. 3.2 and 3.3.

The key result of Ref. [13] is the derivation of a general formula for probabilities associated to the time of an event (such as particle detection) in a general quantum system. The event time t is treated as a coarse-grained, quasi-classical parameter associated to macroscopic records of observation.

Let \mathcal{H} be the Hilbert associated to the physical system under consideration. We incorporate the measurement device into the quantum description, so \mathcal{H} describes the degrees of freedom of both the microscopic particles and the macroscopic measurement apparatus. We assume that \mathcal{H} splits into two subspaces: $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. The subspace \mathcal{H}_+ describes the accessible states of the system given that a specific event is realized; the subspace \mathcal{H}_- is the complement of \mathcal{H}_+ . For example, if the quantum event under consideration is a detection of a particle by a macroscopic apparatus, the subspace \mathcal{H}_+ corresponds to all accessible states of the apparatus given that a detection event has been recorded. We denote the projection operator onto \mathcal{H}_+ as \hat{P} and the projector onto \mathcal{H}_- as $\hat{Q} := 1 - \hat{P}$.

Once the transition has taken place, it is possible to measure the values of a microscopic variable through its correlation to a pointer variable of the measurement apparatus. We denote by \hat{P}_λ projection operators (or, more generally, positive operators) corresponding to different values λ

of a physical magnitude that can be measured only if the quantum event under consideration has occurred. For example, when considering transitions associated to particle detection, the projectors \hat{P}_λ may be correlated to properties of the microscopic particle, such as position, momentum and energy. The set of projectors \hat{P}_λ is exclusive ($\hat{P}_\lambda \hat{P}_{\lambda'} = 0$, if $\lambda \neq \lambda'$). It is also exhaustive given that the event under consideration has occurred; i.e., $\sum_\lambda \hat{P}_\lambda = \hat{P}$.

We also assume that the system is initially ($t = 0$) prepared at a state $|\psi_0\rangle \in \mathcal{H}_+$, and that the time evolution is governed by the self-adjoint Hamiltonian operator \hat{H} .

In Ref. [13], we derived the probability amplitude $|\psi; \lambda, [t_1, t_2]\rangle$ that, given an initial state $|\psi_0\rangle$, a transition occurs at some instant in the time interval $[t_1, t_2]$ and a recorded value λ is obtained for some observable:

$$|\psi_0; \lambda, [t_1, t_2]\rangle = -ie^{-i\hat{H}T} \int_{t_1}^{t_2} dt \hat{C}(\lambda, t) |\psi_0\rangle. \quad (20)$$

where the *class operator* $\hat{C}(\lambda, t)$ is defined as

$$\hat{C}(\lambda, t) = e^{i\hat{H}t} \hat{P}_\lambda \hat{H} \hat{S}_t, \quad (21)$$

and

$$\hat{S}_t = \lim_{N \rightarrow \infty} (\hat{Q} e^{-i\hat{H}t/N} \hat{Q})^N \quad (22)$$

is the restriction of the propagator into \mathcal{H}_-

If $[\hat{P}, \hat{H}] = 0$, i.e., if the Hamiltonian evolution preserves the subspaces \mathcal{H}_\pm , then $|\psi_0; \lambda, t\rangle = 0$. For a Hamiltonian of the form $\hat{H} = \hat{H}_0 + \hat{H}_I$, where $[\hat{H}_0, \hat{P}] = 0$, and H_I a perturbing interaction, to leading order in the perturbation

$$\hat{C}(\lambda, t) = e^{i\hat{H}_0 t} \hat{P}_\lambda \hat{H}_I e^{-i\hat{H}_0 t}. \quad (23)$$

The benefit of Eq. (23) is that it does not involve the restricted propagator \hat{S}_t , which is difficult to compute in practice.

The amplitude Eq. (20) squared defines the probability $p(\lambda, [t_1, t_2])$ that at some time in the interval $[t_1, t_2]$ a detection with outcome λ occurred

$$P(\lambda, [t_1, t_2]) := \langle \psi_0; \lambda, [t_1, t_2] | \psi_0; \lambda, [t_1, t_2] \rangle = \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' \text{Tr}[\hat{C}(\lambda, t) \hat{\rho}_0 \hat{C}^\dagger(\lambda, t)], \quad (24)$$

where $\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0|$.

However, the expression $P(\lambda, [t_1, t_2])$ does not correspond to a well-defined probability measure, because it fails to satisfy the Kolmogorov additivity condition for probability measures

$$P(\lambda, [t_1, t_3]) = P(\lambda, [t_1, t_2]) + P(\lambda, [t_2, t_3]). \quad (25)$$

Eq. (25) does not hold for generic choices of t_1, t_2 and t_3 . However, in a macroscopic system (or in a system with a macroscopic component) one expects that Eq. (25) holds with a good degree of approximation, given a sufficient degree of coarse-graining [16, 17]. Thus, if the time of transition is associated to a macroscopic measurement record, there exists a coarse-graining time-scale σ , such that Eq. (25) holds, for $|t_2 - t_1| \gg \sigma$ and $|t_3 - t_2| \gg \sigma$. Then, Eq. (24) does define a probability measure when restricted to intervals of size larger than σ .

Eq. (24) simplifies if there is a separation of time scales between the macroscopic scale of observation, the coarse-graining time scale and the characteristic timescales of the microscopic

system. In this case, it can be shown [13] that the probabilities Eq. (24) are obtained from the probability density

$$P(\lambda, t) = \int d\tau \text{Tr} \left[\hat{C}(\lambda, t + \frac{\tau}{2}) \hat{\rho}_0 \hat{C}^\dagger(\lambda, t - \frac{\tau}{2}) \right] \quad (26)$$

The probability density Eq. (26) is of the form $\text{Tr}[\hat{\rho}_0 \hat{\Pi}(\lambda, t)]$, where

$$\hat{\Pi}(\lambda, t) = \int d\tau \hat{C}^\dagger(\lambda, t - \frac{\tau}{2}) \hat{C}(\lambda, t + \frac{\tau}{2}). \quad (27)$$

The positive operator

$$\hat{\Pi}_\tau(N) = 1 - \int_0^\infty dt \int d\lambda \hat{\Pi}_\tau(\lambda, t), \quad (28)$$

corresponds to the alternative N that no detection took place in the time interval $[0, \infty)$. The positive operator $\hat{\Pi}_\tau(N)$ together with the positive operators Eq. (27) define a Positive-Operator-Valued Measure (POVM). The POVM Eq. (27) determines the probability density that a transition took place at time t , and that the outcome λ for the value of an observable has been recorded.

3.2 Temporal probabilities from field correlation functions

Next, we employ Eq. (26) for constructing probabilities associated to time-of-arrival measurements in a quantum field theoretic set-up. The system under consideration consists of a quantum field $\hat{\phi}$ that describes microscopic particles and a macroscopic particle detector. Eventually, we will identify the quantum field with the scalar field $\hat{\phi}(x)$, Eq. (18), associated to tunneling, but the results obtained in Sec. 3.2 apply for any quantum field theory. With small modifications, the method applies also to spinor and vector fields. The derivation presented here is an adaptation of the method described in Ref. [13].

The Hilbert space for the total system is the tensor product $\mathcal{F} \otimes \mathcal{H}_a$. The Hilbert space \mathcal{H}_a describes the apparatus's degrees of freedom and \mathcal{F} is the Hilbert space associated to the field $\hat{\phi}$.

The Hamiltonian of the total system is the sum $\hat{H}_0 \otimes 1 + 1 \otimes \hat{H}_a + \hat{H}_I$. \hat{H}_0 is the Hamiltonian that describes the quantum field $\hat{\phi}$, \hat{H}_a describes the dynamics of the apparatus and \hat{H}_I is an interaction term.

The requirements of Lorentz covariance and unitarity in quantum field theory imply that the interaction Hamiltonian \hat{H}_I must be a local functional of the fields $\hat{\phi}(x)$ and $\hat{\phi}^\dagger(x)$. Therefore, we consider an interaction Hamiltonian of the form

$$\hat{H}_I = \int dx \left[\hat{\phi}(x) \otimes \hat{J}^\dagger(x) + \hat{\phi}^\dagger(x) \otimes \hat{J}(x) \right], \quad (29)$$

where \hat{J} and \hat{J}^\dagger are current operators on the Hilbert \mathcal{H}_a of the detector. The interaction Hamiltonian Eq. (29) corresponds to particle detection by absorption.

We also assume an initial state $|\Psi_0\rangle$ of the detector such that

$$\hat{J}(x)|\Psi_0\rangle = 0. \quad (30)$$

This condition guarantees that the detector is sensitive to particles rather than anti-particles. To see this, note that the term $\hat{\phi}^\dagger(x) \otimes \hat{J}(x)$ in the Hamiltonian Eq. (29) corresponds to the processes

of either a particle being created at the measurement device or an anti-particle being annihilated. The second process is negligible if the initial state contains only particles at energies $E < m$. The first process is not desirable in a particle detector. Eq. (30) guarantees that the only transitions in the detector are caused by the interaction with microscopic particles.

Next, we specify the macroscopic variables associated to particle detection. These correspond to degrees of freedom of the macroscopic apparatus and they are expressed in terms of the positive operators $1 \otimes \hat{\Pi}_L$ on $\mathcal{F}(H_1 \oplus \bar{H}_1) \otimes \mathcal{H}_a$, labeled by the values L of a macroscopic observable correlated to a particle's position in the laboratory frame of reference. The operators $\hat{\Pi}_L$ are defined on \mathcal{H}_a and they satisfy the completeness relation $\int dL \hat{\Pi}_L = \hat{P}$, where \hat{P} is the projector onto the subspace \mathcal{H}_+ . We choose $\hat{\Pi}_L$ so that $\sqrt{\hat{\Pi}_L}$ corresponds to an unsharp Gaussian sampling of position at $X = L$,

$$\sqrt{\hat{\Pi}_L} = \frac{1}{(\pi\delta^2)^{1/4}} \sum_a \int dX e^{-\frac{(L-X)^2}{2\delta^2}} |X, a\rangle \langle X, a|, \quad (31)$$

The index a in Eq. (31) refers to the degrees of freedom of the apparatus other than the pointer variable, and δ corresponds to the spatial resolution of the detector.

We place no restriction on the apparatus' Hamiltonian \hat{H}_a , except for the requirement that it commutes with the operator \hat{P} : $[\hat{H}_a, \hat{P}] = 0$. This implies that the self-evolution of the apparatus degrees of freedom does not lead to spurious detection records. It follows that $[1 \otimes \hat{P}, \hat{H}_s \otimes 1 + 1 \otimes \hat{H}_a] = 0$; hence, the class operators $\hat{C}(\lambda, t)$ follow from Eq. (23).

Substituting into Eq. (27), we obtain

$$P(L, t) = \int d\tau \int dx dx' \langle \psi_0 | \hat{\phi}^\dagger(x', t - \tau/2) \hat{\phi}(x, t + \tau/2) | \psi_0 \rangle R(L, \tau, x, x') \quad (32)$$

where $\hat{\phi}(x, t)$ and $\hat{\phi}^{dagger}(x, t)$ are the Heisenberg-picture fields, $|\psi_0\rangle$ is the initial field state and

$$R(L, \tau, x, x') = \langle \Psi_0 | \hat{J}(x') \sqrt{\hat{\Pi}_L} e^{i\hat{H}_a\tau} \sqrt{\hat{\Pi}_L} \hat{J}^\dagger(x) | \Psi_0 \rangle. \quad (33)$$

is a kernel that involves the detailed physics of the detector.

The macroscopic record of a detection must be correlated with the locus of the interaction, in the sense that $\sqrt{\hat{\Pi}_L} \hat{J}^\dagger(x) | \Psi_0 \rangle \simeq 0$ if $|L - x|$ is much larger than the detector's resolution δ . This implies that x and x' in Eq. (33) lie within a distance δ from L , otherwise the kernel $R(L, \tau, x, x')$ vanishes. Equivalently, the kernel R vanishes unless $X = (x + x')/2$ lies within distance of order δ from L , and $|z|$ is of order δ . If δ is much smaller than the distance between source and detector, we can approximately set $L = (x + x')/2$, so that

$$R(L, \tau, x, x') \simeq \delta \left(L - \frac{x + x'}{2} \right) g(x - x', \tau), \quad (34)$$

This form for $R(L, \tau, x, x')$ is consistent with several different detector models that have been constructed in Ref. [13]. The kernel $g(z, \tau)$ depends on the detailed physics of the detector. It vanishes for values of $|z|$ larger than the position resolution δ of the detector and $|\tau|$ larger than the decoherence time of the detector, respectively.

Eq. (32) becomes

$$P(L, t) = \int d\tau \int dz \langle \psi_0 | \hat{\phi}^\dagger(L - \frac{z}{2}, t - \frac{\tau}{2}) \hat{\phi}(L + \frac{z}{2}, t + \frac{\tau}{2}) | \psi_0 \rangle g(z, \tau). \quad (35)$$

It is important to observe that no assumption about the form of the Hamiltonian \hat{H}_0 , or about the initial state $|\psi_0\rangle$ has been made for the derivation of Eq. (32) or Eq. (35). Eq. (32) reveals a relation between detection probability and field correlation functions that applies to any quantum field theory. The probability of detection is determined by the two-point function of the fields that interact with the detectors.

3.3 The probability distribution for the time of detection

Next, we specialize to the case of the fields $\hat{\phi}(x, t)$, $\hat{\phi}^\dagger(x, t)$ of Eqs. (18–19), associated to a tunneling set-up. We consider a single-particle initial state for the field $|\psi_0\rangle = \int dx \hat{a}^\dagger(x) \psi_0(x) |0\rangle$, where $|0\rangle$ is the field vacuum and $\psi_0(x)$ a single-particle wave function concentrated on positive values of momentum.

Substituting Eqs. (18–19) into Eq. (35), we obtain

$$P(L, t) = P_0(L, t) + P_1(t) + P_2(L, t), \quad (36)$$

where

$$\begin{aligned} P_0(L, t) &= \int d\tau \int dz g(z, \tau) \left[\int dx \Delta(L + \frac{z}{2}, x; \tau + \frac{\tau}{2}) \psi_0(x) \right] \\ &\times \left[\int dx' \Delta(L - \frac{z}{2}, x'; \tau - \frac{\tau}{2}) \psi_0(x') \right]^* \end{aligned} \quad (37)$$

$$P_1(L, t) = \int d\tau \int dz g(z, \tau) \sum_{\sigma} \left(\int_{-\infty}^{\infty} \frac{dk}{(2\pi)(2E_k)} e^{-ikz + iE_k\tau} \right) \quad (38)$$

$$P_2(L, t) = \int d\tau \int dz g(z, \tau) D(L - \frac{z}{2}, L + \frac{z}{2}; -\tau), \quad (39)$$

and

$$\Delta(x, x'; t) = \sum_{\sigma=-}^{+} \int_0^{\infty} \frac{dk}{\sqrt{2E_k}} f_{k\sigma}(x) f_{k\sigma}^*(x') e^{-iE_k t} \quad (40)$$

$$D(x, x'; t) = \sum_n \frac{1}{2E_n} g_n^*(x) g_n(x') e^{-iE_n t}. \quad (41)$$

The terms $P_1(L, t)$ and $P_2(L, t)$ are independent of the initial state of the particle. They correspond to ‘false alarms’, that is spurious detection events—such terms are generic in the theory of relativistic quantum measurements [31]. They act as noise that obscures the signal term $P_0(L, t)$. The P_1 term is independent on the location of the detector. It is also independent of the barrier, as it persists for $V(x) = 0$. Its contribution to the total probability is negligible for detector sizes much larger than the de-Broglie wave-length of particles. The term P_2 originates from the anti-particle bound states and it depends on the position L of the detector. However, the function $D_2(x, x', t)$, Eq. (41) decays exponentially outside the barrier region, with a rate given by γ_n , Eq. (15). Therefore, the term $P_2(L, t)$ is strongly suppressed for $L \gg \max_n \{\gamma_n^{-1}\}$. The largest values of γ_n are of the order of the length $\sqrt{d/m} \ll d$.

For a detector far from the barrier ($L \gg d$), the contributions from the false-alarm terms P_1 and P_2 becomes negligible, and $P(L, t) = P_0(L, t)$.

The term $P_0(L, t)$ involves the function $\Delta(x, x'; t)$, Eq. (41). For $x > \frac{d}{2}$, $x' < -\frac{d}{2}$, we obtain

$$\begin{aligned} \Delta(x, x'; t) &= \int_0^{\infty} \frac{dk}{(2\pi)\sqrt{2E_k}} \left[e^{ik(x-x')} (T_k + w_k \frac{-R_k + w_k T_k}{1 - |w_k|^2}) + e^{-ik(x+x')} \frac{-w_k}{1 - w_k^2} \right. \\ &\left. + e^{ik(x+x')} \frac{R_k^* + (T_k^* - w_k R_k^*)(R_k - w_k T_k)}{1 - w_k^2} + e^{-ik(x-x')} \frac{T_k^* - w_k R_k^*}{1 - |w_k|^2} \right] e^{-iE_k t}. \end{aligned} \quad (42)$$

The term in the second line of Eq. (42) vanishes when $\Delta(x, x'; t)$ acts on wave functions with support on positive values of momentum. The second term in the first line of Eq. (42) is strongly suppressed for $x \simeq L$ and $t > 0$.

Thus, for sufficiently large L , the probability density $P(L, t)$ becomes

$$P(L, t) = \int_0^\infty \frac{dk}{(2\pi)\sqrt{2E_k}} \int_0^\infty \frac{dk'}{(2\pi)\sqrt{2E_{k'}}} f(k, k') A_k A_{k'}^* \tilde{\psi}_0(k) \tilde{\psi}_0(k') e^{i(k-k')L - i(E_k - E_{k'})t}, \quad (43)$$

where

$$A_k = \frac{T_k - w_k R_k}{1 - w_k^2}, \quad (44)$$

$$f(k, k') = \int d\tau \int dz g(\tau, z) e^{i\frac{k+k'}{2}z + i\frac{E_k + E_{k'}}{2}\tau}, \quad (45)$$

$\tilde{\psi}_0(k)$ is the Fourier transform of the initial state, and $E_k = \sqrt{k^2 + m^2}$.

For an initial state with support on positive momenta, $P(L, t)$ is strongly suppressed for $t < 0$. Hence, we approximate $\int_0^\infty P(L, t) \simeq \int_{-\infty}^\infty P(L, t)$. For a free particle ($A_k = 1$), the time-integrated probability for a particle to be detected at $x = L$ is

$$\int_0^\infty P(L, t) = \int_0^\infty \frac{dk}{2\pi} |\tilde{\psi}_0(k)|^2 \frac{f(k, k)}{2E_k v_k}. \quad (46)$$

The quantity $\alpha(k) = \frac{f(k, k)}{2E_k v_k}$ then defines the absorption coefficient of the detector for particles with momentum k , i.e., the fraction of the number of incident particles absorbed by the detector. The function $f(k, k')$ depends on k and k' only through the combinations $(k + k')/2$ and $(E_k + E_{k'})/2$. For an initial state with momentum spread δk much smaller than the mean momentum \bar{k} , we approximate

$$f(k, k') = \sqrt{|f(k, k)|} \sqrt{|f(k', k')|}. \quad (47)$$

Then, Eq. (43) becomes

$$P(L, t) = |\mathcal{A}(L, t)|^2. \quad (48)$$

where

$$\mathcal{A}(L, t) = \int_0^\infty \frac{dk}{2\pi} \sqrt{\alpha(k) |v_k|} A_k \tilde{\psi}_0(k) e^{ikL - iE_k t} \quad (49)$$

Eqs. (48—49) are the main result of this paper. They express the probability distribution for particle detection in a tunneling experiments in terms of the transmission and reflection coefficients of the barrier, encoded in the coefficient A_k . All information about the detector is encoded in the absorption coefficient $\alpha(k)$ for particles of momentum k and all information about the barrier in the complex amplitude A_k .

In absence of the potential barrier $A_k = 1$, and Eq. (48) reduces to the time-of-arrival distribution for free relativistic particles derived in Ref. [13], which generalizes Kijowski's probability distribution for the time of arrival of non-relativistic particles [30].

The time-integrated probability Eq. (48) is

$$\int_{-\infty}^\infty dt P(L, t) = \int \frac{dk}{2\pi} \alpha(k) |A_k|^2 |\tilde{\psi}_0(k)|^2. \quad (50)$$

Eq. (50) implies that $|A_k|^2$ is the transmission probability, i.e., the probability that a particle of momentum k crosses the barrier at any time t . For parity symmetric potentials, $A_k = T_k$ and the transmission probability coincides with the standard expression $|T_k|^2$. In general, however, they differ. The reason is that in the QFT description of tunneling, both right-moving and left-moving modes contribute to field correlation functions, and by virtue of Eq. (35), they contribute to the detection probability.

It is convenient to parameterize $T_k = |T_k|e^{i\phi_k}$ and $R_k = -i|R_k|e^{i\phi_k+i\chi_k}$. In parity-symmetric potentials, $\chi_k = 0$. In the regime of opaque barrier, $|T_k| \ll |R_k| \simeq 1$, hence, to leading order in $|T_k|$

$$A_k \simeq \frac{1}{2}|T_k|(1 + e^{2i\chi_k}). \quad (51)$$

We then obtain

$$|A_k|^2 \simeq |T_k|^2 \cos^2 \chi_k \quad (52)$$

For potentials characterized by large deviations from parity symmetry, the transmission probability $|A_k|^2$ may differ significantly from $|T_k|^2$. In fact, for $\chi_k = \frac{\pi}{2}$, the transmission probability vanishes even if $|T_k|^2$ is non-zero.

3.4 Delay time

Next, we examine the case of an initial state localized around $x = -x_0$ and concentrated around the value $k = p$ for momentum. In particular, we consider a wave function $\tilde{\psi}_0(k) = \tilde{u}_0(k - p)e^{ikx_0}$, where $\tilde{u}_0(k)$ is a positive-valued, square-integrable function centered around $k = 0$. Then, the integrand in Eq. (48) involves a rapidly oscillating phase $\exp[ik(x_0 + L) - iE_k t + \theta_k]$, where $\theta_k = Im \log A_k$. Often, but not always, this implies that the integral is strongly suppressed unless the phase is stationary for $k = p$, i.e., unless

$$x_0 + L - v_p t + \theta'_p = 0. \quad (53)$$

In general, the stationary phase approximation is valid if the probability density Eq. (48) has a single peak. In this case, the solution to Eq. (53), $\bar{t}(p) := (x_0 + L + \theta'_p)/v_p$, determines the location of the peak.

In absence of the barrier, and for the same initial state, the time-of-arrival probability distribution is peaked around $\bar{t}_0(p) = (x_0 + L)/v_p$. Thus, when comparing two time-of-arrival experiments, one with the barrier and one without, particles in the former are detected with a delay

$$t_d(p) = \bar{t} - \bar{t}_0(p) = \theta'_p/v_p = \frac{1}{v_k} Im \left[\frac{\partial \log A_k}{\partial k} \right]_{k=p}. \quad (54)$$

when compared with detected particles in the latter. For parity symmetric potentials, $A_k = T_k$ and the delay time Eq. (54), when defined, coincides with the Bohm-Wigner time [26]. In general, they differ.

Given Eq. (54) one may infer that the time particles spent inside the barrier region equals

$$\tau_p = t_d(p) - d/v_p. \quad (55)$$

However, this inference is based on an analogy from classical physics and it does not follow from the rules of quantum theory. The time delay $t_d(p)$, Eq. (54), is obtained from the comparison of probability distributions from different experiments; it is not directly measured in a single

experiment. The delay time is not a quantum observable or a random variable of the theory. It is a temporal parameter that appears in the probability distribution $P(L, t)$ for a specific class of initial states. This parameter may incorporate significant information about properties of the barrier, but only if it captures a significant physical feature of the probability density $P(L, t)$.

Moreover, Eq. (54) follows from a saddle-point approximation to the probability density $P(L, t)$, Eq. (48) that is valid only if the integrand in Eq. (48) has a single maximum. This is not true in general, as we will see in the study of the double square barrier in Sec. 4.3. In such cases, the naive application of the saddle point approximation leads to erroneous physical conclusions. In general, higher moments of the probability distribution $P(L, t)$ contain significant information and the temporal aspects of the tunneling process cannot be adequately described in terms of a single parameter such as the delay time.

In what follows, we will refer to τ_p , Eq. (55), as "tunneling time", for convenience, without committing to its interpretation as the transit time of the particle through the barrier region. We will view τ_p as a useful time parameter that characterizes the probability distribution $P(L, t)$.

4 Application: Piecewise constant potentials

In this section, we construct the probability distribution for the time of detection for two important examples of tunneling potentials: the square barrier and the symmetric double square barrier. Both potentials are parity-symmetric, so the amplitude A_k , Eq. (44), coincides with the transmission amplitude T_k .

We consider an initial state $\tilde{\psi}_0(k)$, centered around $x = -x_0$ and well concentrated around a mean momentum $k = p$,

$$\tilde{\psi}_0(k) = \tilde{u}_0(k - p)e^{ikx_0}, \quad (56)$$

where \tilde{u}_0 is an even, positive-valued wave function centered around $k = 0$ with width σ_p . We assume that the absorption coefficient $\alpha(k)$ does not vary significantly with momentum, so that we can set it equal to unity. Then, Eq. (48) takes the form

$$P(L, t) = \left| \int_0^\infty \frac{dk}{2\pi} \sqrt{|v_k|} T_k \tilde{u}_0(k - p) e^{ik(L+x_0) - iE_k t} \right|^2. \quad (57)$$

In what follows, we evaluate Eq. (57) for the square barrier and the symmetric double square barrier.

4.1 Junction conditions

Explicit calculations of tunneling time are usually performed for piecewise constant potentials. In such potentials, eigenstates of the Hamiltonian are identified from junction conditions on the points of discontinuity. The Hamiltonians h_1 and h_2 , Eq. (10) are non-local operators, and their junction conditions turn out to be different from the ones usually considered in the literature. Here, we present the derivation of the junction conditions.

We consider the Hamiltonian h_1 , with a piecewise constant potential. For piecewise constant potentials, $\tilde{V}(x) = V(x)$ except for the points of discontinuity. Hence, except for the points of discontinuity, the eigenfunctions of h_1 are of the form $Ae^{ikx} + Be^{-ikx}$ where k may be either real or imaginary. Let us assume that $x = 0$ is a point of discontinuity, and consider a neighborhood U around $x = 0$ with no other point of discontinuity. Let $g(x)$ be an eigenstate of h_1 with energy

E . Assuming $x \in U$, $g''(x) = -k_1^2 g(x)$ for $x < 0$, and $g''(x) = -k_2^2 g(x)$ for $x > 0$, where k_1 and k_2 are real or imaginary constants. Then,

$$\int_{-\epsilon}^{\epsilon} dx h_1 \psi(x) = E \int_{-\epsilon}^{\epsilon} dx \psi(x), \quad (58)$$

for some constant ϵ that will be eventually taken to zero. For continuous $g(x)$, all terms in the above equation that do not involve derivatives vanish at the limit $\epsilon \rightarrow 0$. Hence, as $\epsilon \rightarrow 0$,

$$\int_{-\epsilon}^{\epsilon} dx \sqrt{-\partial_x^2 + m^2} g(x) + \int_{-\epsilon}^{\epsilon} dx \delta V g(x) = 0, \quad (59)$$

where $\delta V = [[V, h_0^{1/2}], h_0^{-1/2}]$.

We write $\sqrt{p^2 + m^2} = \sum_{n=0}^{\infty} a_n p^{2n}$, in terms of the coefficients a_n obtained from the binomial expansion. The first term in Eq. (59) becomes

$$\begin{aligned} & \sum_{n=1}^{\infty} a_n \int_{-\epsilon}^{\epsilon} dx (-\partial_x^2)^n g(x) = \sum_{n=1}^{\infty} a_n (-1)^n (\partial_x^{2n-1} g(\epsilon) - \partial_x^{2n-1} g(-\epsilon)) = \\ & - \sum_{n=1}^{\infty} a_n ((k_2)^{2n-2} g'(\epsilon) - (k_1)^{2n-2} g'(-\epsilon)) = - \frac{\sum_{n=1}^{\infty} a_n k_2^{2n}}{k_2^2} g'(\epsilon) + \frac{\sum_{n=1}^{\infty} a_n k_1^{2n}}{k_1^2} g'(-\epsilon) = \\ & - \frac{\sqrt{m^2 + k_2^2} - m}{k_2^2} g'(\epsilon) + \frac{\sqrt{m^2 + k_1^2} - m}{k_1^2} g'(-\epsilon). \end{aligned} \quad (60)$$

Writing $1/\sqrt{p^2 + m^2} = \sum_n b_n p^{2n}$, the operator δV can be expressed as a series

$$\delta V = -V_0 \sum_n a_n \sum_m b_m \sum_{k=0}^{2n} \sum_{l=0}^{2m} \hat{p}^{k+l} \delta'(\hat{x}) \hat{p}^{2n+2m-l-k}, \quad (61)$$

where V_0 is the potential jump at $x = 0$. Consider the integral of a single term in the series above

$$\begin{aligned} \int_{-\epsilon}^{\epsilon} dx \hat{p}^{k+l} \delta'(x) \hat{p}^{2n+2m-k-l} g(x) &= (-i)^{2n+2m} \sum_{r=0}^{k+l} \int_{-\epsilon}^{\epsilon} dx \binom{k+l}{r} \delta^{(1+r)}(x) \partial_x^{2n+2m-r} g(x) = \\ &= (-i)^{2n+2m} \sum_{r=0}^{k+l} \binom{k+l}{r} (-1)^r \int_{-\epsilon}^{\epsilon} dx \delta'(x) \partial_x^{2n+2m} g(x). \end{aligned} \quad (62)$$

The sum over r involves the term $\sum_{r=0}^{k+l} \binom{k+l}{r} (-1)^r = (1-1)^{k+l} = 0$. Hence, the integral above vanishes, and so does the second term in Eq. (59).

Our final result is the junction condition

$$\frac{\sqrt{m^2 + k_2^2} - m}{k_2^2} g'(0_+) = \frac{\sqrt{m^2 + k_1^2} - m}{k_1^2} g'(0_-), \quad (63)$$

which is to be implemented together with the continuity condition

$$g(0_+) = g(0_-) \quad (64)$$

For $|k_1| \ll m$ and $|k_2| \ll m$, Eq. (63) reduces to the non-relativistic junction condition $g'(0_+) = g'(0_-)$.

4.2 Square barrier

4.2.1 Transmission amplitude

Next, we consider the square barrier potential,

$$V(x) = \begin{cases} V_0, & x \in [-\frac{d}{2}, \frac{d}{2}] \\ 0, & x \notin [-\frac{d}{2}, \frac{d}{2}] \end{cases} \quad (65)$$

For $E - V_0 < m$, we apply the junction conditions (63) at $x = \pm \frac{d}{2}$, to obtain the transmission and reflection coefficients

$$T_k = \frac{e^{-ikd}}{\cosh(\lambda_k d) - i\eta_k \sinh(\lambda_k d)} \quad (66)$$

$$R_k = -i \frac{e^{-ikd} \rho_k \sinh(\lambda_k d)}{\cosh(\lambda_k d) - i\eta_k \sinh(\lambda_k d)} \quad (67)$$

where

$$\lambda_k = \sqrt{m^2 - (E - V_0)^2}, \quad (68)$$

and the functions

$$\eta_k = \frac{1}{2} \left(e_k - \frac{1}{e_k} \right) \quad (69)$$

$$\rho_k = \frac{1}{2} \left(e_k + \frac{1}{e_k} \right) \quad (70)$$

are expressed in terms of the quantity

$$e_k = \frac{\lambda_k}{k} \frac{\sqrt{m^2 + k^2} - m}{m - \sqrt{m^2 - \lambda_k^2}}. \quad (71)$$

In the non-relativistic limit, $e_k = k/\lambda_k$ and Eqs. (66–67) reduce to the standard textbook expressions for the square barrier potential.

4.2.2 Tunneling time

Expressing the transmission amplitude (66) as $T_k = |T_k|e^{i\phi_k}$, we note that the phase ϕ_k varies much faster with k than $|T_k|$. Hence, when computing the probability density Eq. (57) for a sufficiently narrow initial wave-packet, we may assume that $|T_k| \simeq |T_p|$. The velocity v_k varies slowly with k in comparison to the phases, hence, we also set $v_k = v_p$. Expanding the phase factor to first order in $k - p$, $\phi_k = \phi_p + \phi'_p(k - p)$, we obtain

$$P(L, t) = v_p |T_p|^2 |u_0[v_p t - (L + x_0 + \phi'_p)]|^2, \quad (72)$$

where $u_0(x) = \int \frac{dk}{2\pi} e^{-ikx} \tilde{u}_0(k)$ is the wave function u_0 in the position representation. The probability density Eq. (72) exhibits a single peak at $t = (L + x_0 + \phi'_p)/v_p$, thus leading to the identification of delay time $t_d(p) = \phi'_p/v_p$, in accordance with Eq. (54). Using Eq. (66), we obtain

$$t_d(p) = -\frac{d}{v_p} + \tau_p, \quad (73)$$

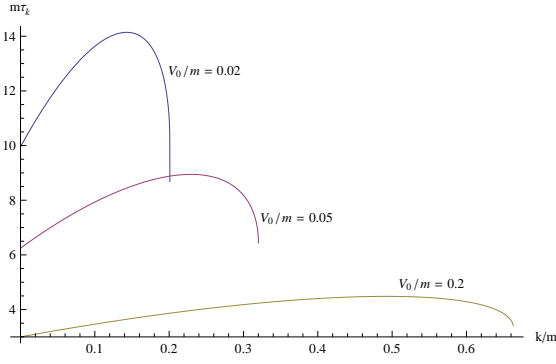


Figure 1: Tunneling times τ_p as a function of the incoming momentum $p \in [0, \sqrt{2mV_0 + V_0^2}]$, for different values of the potential V_0 and barrier length $d = 5000m^{-1}m$, m the particle's mass.

where the tunneling time τ_k is

$$\tau_p = \frac{-\eta_p \frac{\sqrt{m^2 - \lambda_p^2}}{\lambda_p} d + m\rho_p \left(\frac{1}{p} + \frac{1}{\lambda_p}\right) \sinh(\lambda_p d) \cosh(\lambda_p d)}{\cosh^2(\lambda_p d) + \eta_p^2 \sinh^2(\lambda_p d)} \quad (74)$$

Eq. applies to all values of p corresponding to tunneling: $0 \leq p \leq \sqrt{(m + V_0)^2 - m^2}$.

For a long barrier $e^{-\lambda_k d} \ll 1$,

$$T_k = 2 \frac{e^{-ikd} e^{-\lambda_k d}}{1 - i\eta_k}, \quad (75)$$

and the tunneling time is independent of d

$$\tau_p = m \frac{\rho_p}{1 + \eta_p^2} \left(\frac{1}{p} + \frac{1}{\lambda_p} \right). \quad (76)$$

4.3 Double square barrier

Next, we consider the case of the symmetric double square barrier, that is, the piecewise constant potential

$$V(x) = \begin{cases} 0 & x \in (-\infty, -a - \frac{r}{2}) \cup (-\frac{r}{2}, \frac{r}{2}) \cup (a + \frac{r}{2}, \infty) \\ V_0 & x \in [-a - \frac{r}{2}, -\frac{r}{2}] \cup [\frac{r}{2}, a + \frac{r}{2}] \end{cases}. \quad (77)$$

This potential corresponds to two identical square barriers of width a separated by a distance r .

4.3.1 The transmission amplitude

We compute the transmission probability by implementing the junction conditions (63) at $x = \pm(a + \frac{r}{2})$ and at $x = \pm\frac{r}{2}$. We find,

$$T_k = \frac{e^{-ik(r+2a)}}{(\cosh(\lambda_k a) - i\eta_k \sinh(\lambda_k a))^2 e^{-ikr} + \rho_k^2 \sinh^2(\lambda_k a) e^{ikr}}, \quad (78)$$

where λ_k , η_k and ρ_k are functions of k that refer to a single barrier and they are defined in Eqs. (68—70).

The transmission amplitude Eq. (78) can also be expressed as

$$T_k = \frac{T_{0k}^2}{1 - R_{0k}^2 e^{2ik(r+a)}}, \quad (79)$$

where T_{0k} and R_{0k} stand for the transmission and reflection amplitudes for the *single* barrier, as given by Eqs. (66) and (67) respectively.

Writing $T_{0k} = |T_{0k}|e^{i\phi_k}$ and $R_{0k} = -i|R_{0k}|e^{i\phi_k}$, Eq. (79) becomes

$$T_k = \frac{T_{0k}^2}{1 + |R_{0k}|^2 e^{2i[k(r+a)+\phi_k]}} \quad (80)$$

Eq. (80) implies that for $\cos[2(k(r+a) + \phi_k)] = -1$, $|T_k| = 1$. It follows that any momenta $0 \leq k_n \leq \sqrt{V_0^2 + 2mV_0}$ that are solutions to the equation

$$2k_n(r+a) + \phi_{k_n} = (2n+1)\pi, \quad n = 0, 1, 2, \dots \quad (81)$$

correspond to *resonances* of the double-barrier potential.

4.3.2 Large inter-barrier separation

Next, we expand the denominator of the transmission amplitude Eq. (80) as a geometric series,

$$T_k = |T_{0k}|^2 \sum_{n=0}^{\infty} (-1)^n |R_{0k}|^{2n} e^{2i[nk(R+a) + (n+1)\phi_k]}. \quad (82)$$

Substituting into Eq. (57), we find that the probability density $P(L, t)$ involves the amplitude

$$\mathcal{A}(L, t) = \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} dk \sqrt{v_k} \tilde{u}_0(k-p) |T_{0k}|^2 |R_{0k}|^{2n} e^{ik(x_0+L) - iE_k t + 2i[nk(R+a) + (n+1)\phi_k]}. \quad (83)$$

We evaluate the integral Eq. (83) using the same approximations as in Sec. 4.2.2. We set $|T_{0k}| = |T_{0p}|$, $|R_{0k}| = |R_{0p}|$, $v_k = v_p$, and we expand the phase factor to first order in $k-p$. Taking the limits of integration to $k \in (-\infty, \infty)$, we obtain

$$\begin{aligned} \mathcal{A}(L, t) &= \sqrt{v_p} |T_{0p}|^2 e^{ipL - iE_p t} \\ &\times \sum_{n=0}^{\infty} (-1)^n |R_{0p}|^{2n} e^{2i[np(R+a) + (n+1)\phi_p]} u_0[(x_0 + L) - v_p t + 2\phi'_p + 2n(r + v_p \tau_p)], \end{aligned} \quad (84)$$

where τ_p is the tunneling time, Eq. (55), and $u_0(x)$ is the wave function \tilde{u}_0 transformed to the position representation.

The amplitude Eq. (85) involves a sum of terms labeled by n , with each term peaked around time

$$t_n = [x_0 + L + \phi'_p + 2n(r + v_p \tau_p)]/v_p. \quad (85)$$

The distance between two successive peaks is

$$\Delta t = 2(r/v_p + \tau_p). \quad (86)$$

If the position spread $\sigma_x > 1/(2\sigma_p)$ of the wave-function u_0 satisfies $\sigma_x \ll v_p \Delta t$, then there is no overlap between the terms in the sum Eq. (85). Hence, the amplitude Eq. (85) is close to zero

except for neighborhoods of width σ_x/v_p peaked around the values t_n , Eq. (85). The first peak in the amplitude is at time

$$t_0 = (L + x_0)/v_p + 2\phi'_p/v_p. \quad (87)$$

The next term in the series Eq. (85) has a peak at $t_0 + \Delta t$. The delay by a factor Δt can be attributed to the fraction of particles transmitted through the first barrier, reflected at the second barrier, reflected at the first barrier and exiting the second barrier in its second attempt. Similarly the peak at time t_n corresponds to particles that were transmitted through the first barrier, they were subsequently reflected n times at the second barrier and n times at the first barrier, and finally exited after their $(n + 1)$ -th attempt on the second barrier. The attempts of crossing the barrier are statistically independent: a particle exiting in its $(n + 1)$ -th attempt has been reflected $2n$ times. Since the amplitude for a single reflection is R_{0p} , the contribution of the successful exit by the $(n + 1)$ -th attempt is suppressed by a factor $|R_{0p}|^{2n}$.

The detection probability $P(L, t) = |\mathcal{A}(L, t)|^2$ is

$$P(L, t) = v_p |T_{0p}|^4 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |R_{0p}|^{2(n+m)} e^{i(n-m)\beta_p} u_0(v_p \Delta t [n - (t - t_0)/\Delta t]) u_0(v_p \Delta t [m - (t - t_0)/\Delta t]), \quad (88)$$

where

$$\beta_p = (2p(r + a) + 2\phi_p + \pi) \mod (2\pi) \quad (89)$$

For $\sigma_x \ll v_p \Delta t$, the function u_0 in Eq. (88) approximates well a delta function. Delta functions satisfy the property $\delta(x)\delta(y) = \delta(\frac{x+y}{2})\delta(x-y)$. We expect that an analogous equality also holds approximately for smeared delta functions, i.e., for families of functions $F_a(x)$, such that $\lim_{a \rightarrow 0} F_a(x) = \delta(x)$, when a is close to zero. Hence, we can write $u_0(x)u_0(y) = u_0[\frac{1}{2}(x+y)]u_0(x-y)$ in Eq. (88); the equality is *exact* for Gaussian wave-functions. We also rearrange the double sum in Eq. (88) as $\sum_{N=0}^{\infty} \sum_{M=-N}^N$, where $N = m + n$ and $N = n - m$. Then, the time dependence of Eq. (88) is contained in a term

$$\sum_{N=0}^{\infty} |R_{0p}|^{2N} u_0(v_p \Delta t [\frac{1}{2}N - (t - t_0)/\Delta t]). \quad (90)$$

Thus, the probability distribution $P(L, t)$ is an infinite sum of terms, each sharply peaked at times $t_N = t_0 + \frac{1}{2}N\Delta t$ and suppressed by a factor $|R_{0p}|^{2N}$. If the time-scale of observation is much larger than Δt and $|T_{0p}|^2 \ll 1$, we can approximate the probability distribution by a smooth function connecting the peaks of Eq. (90), modulo an overall normalization constant. In this approximation, the probability distribution $P(L, t)$ vanishes for $t < t_0$. For $t > t_0$, $P(L, t)$ is proportional to $|R_{p0}|^{2N} = e^{N \log(1-|T_{0p}|^2)} \simeq e^{-N|T_{0p}|^2}$, where $N = 2(t - t_0)/\Delta t$.

Hence,

$$P(L, t) = \begin{cases} 0 & t < t_0 \\ C e^{-\Gamma_p(t-t_0)} & t > t_0 \end{cases}, \quad (91)$$

where

$$\Gamma_p = \frac{2|T_{0p}|^2}{\Delta t} = \frac{|T_{0p}|^2}{\frac{r}{v_p} + \tau_p}. \quad (92)$$

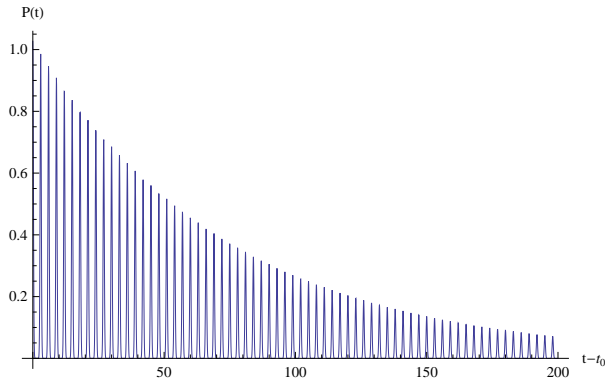


Figure 2: The probability density $P(L, t)$, Eq. (88), for particle detection for the double square barrier potential, as a function of the time $t - t_0$ after first detection. The probability density is a sum of peaks separated by distance Δt . The peaks in the distribution decay exponentially with rate Γ_p , Eq. (92).

is a decay coefficient associated to the barrier, and C is a positive constant.

Eq. (91) applies for (i) p far from a resonance frequency of the double barrier system and (ii) for wave-packets with position spread much smaller than $v_p \Delta t$.

The coefficient C Eq. (91) is determined by the requirement that the time-integrated probability is consistent with Eq. (50), i.e., the requirement that

$$\int_0^\infty P(L, t) dt \simeq |T_p|^2 = \frac{|T_{0p}|^4}{1 + |R_{0p}|^4 + 2|R_{0p}|^2 \cos \beta_p}. \quad (93)$$

This leads to

$$P(L, t) = \begin{cases} 0 & t < t_0 \\ |T_p|^2 \Gamma_p e^{-\Gamma_p(t-t_0)} & t > t_0 \end{cases}, \quad (94)$$

The physical interpretation of Eq. (94), expressed in classical language, is the following. The first signal arrives at the detector at a time t_0 which corresponds to a delay time $t_d(p) = 2\phi'_p/v_p$ that is twice the delay time of the single barrier. The associated tunneling time τ_p^{2b} for the particles at first detection is

$$\tau_p^{2b} = t_d(p) + \frac{r + 2a}{v_p} = 2\tau_p + \frac{r}{v_p}. \quad (95)$$

The classical interpretation of Eq. (95) is that the particles cross the two barriers independently and at time equal to the tunneling time τ_p of each barrier. In the inter-barrier region, the particles propagate freely with velocity v_p . Particles crossing the first barrier, are trapped in the inter-barrier region, and escape only after multiple attempts to cross the second barrier. The detection probability of those later-escaping particles decays exponentially with a rate Γ_p .

4.3.3 Wave-packets narrowly concentrated in momentum

In Sec. 4.3.3, we considered the regime $\sigma_p v_p \Delta t \gg 1$ that corresponds to an inter-barrier distance r much larger than the spread of the initial wave-packet. In the opposite regime, of small inter-barrier distance r , the peaks in the amplitude Eq. (85) essentially overlap. Thus, it is a meaningful approximation to substitute the infinite sum in Eq. (85) with an integral over a continuous variable, using the Euler-MacLaurin summation formula.

We consider a Gaussian $\tilde{u}_0(k)$,

$$\tilde{u}_0(k) = \left(\frac{4\pi}{\sigma_p^2}\right)^{1/4} e^{-\frac{k^2}{4\sigma_p^2}}. \quad (96)$$

Then, Eq. (85) becomes

$$\mathcal{A}(L, t) \simeq \sqrt{v_p} |T_{0p}|^2 \left(\frac{\sigma_p^2}{\pi}\right)^{1/4} e^{i(pL - E_p t + 2\phi_p)} \left[\int_0^\infty dn e^{-\sigma_p^2 v_p^2 \Delta t^2 \left(n - \frac{t-t_0}{\Delta t}\right)^2 - n(|T_{0p}|^2 - i\beta_p)} + \frac{1}{2} e^{-\sigma_p^2 v_p^2 (t-t_0)^2} \right], \quad (97)$$

where t_0 is given by Eq. (87), Δt is given by Eq. (86) and β_p by Eq. (89).

For $t < t_0$, the amplitude Eq. (97) is strongly suppressed. In the vicinity of $t = t_0$, small values of n dominate in the integral Eq. (97), so we can ignore the contribution of the term $e^{-n|T_{0p}|^2}$. Then,

$$P(L, t) = v_p |T_{0p}|^4 \left(\frac{\sigma_p^2}{\pi}\right)^{1/2} \left| \frac{\sqrt{\pi}}{2\sigma_p v_p \Delta t} e^{-\mu_p^2} \text{erfc}(-\sigma_p v_p (t - t_0) + i\mu_p) + \frac{1}{2} e^{-\sigma_p^2 v_p^2 (t-t_0)^2} \right|^2, \quad (98)$$

where $\mu_p = \frac{\beta_p}{2\sigma_p v_p \Delta t}$, and erfc stands for the complementary error function

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \quad (99)$$

The probability density (98) is strongly peaked around $t = t_0$, hence, also in this regime, the delay time for the double barriers is twice the delay time of the single barrier.

At later times ($t - t_0 \gg 1/(\sigma_p v_p)$), we can take the limit of integration in Eq. (97) to $-\infty$, and drop the term outside the integral. Then,

$$P(L, t) = K e^{-\Gamma_p (t-t_0)}, \quad (100)$$

where

$$K = \frac{\sqrt{\pi} \Gamma_p^2}{4v_p \sigma_p} e^{\frac{\Gamma_p^2}{16} - \mu_p^2} \quad (101)$$

is a constant.

Hence, following a transient period of order $1/(\sigma_p v_p)$ after the first particle detection, the probability $P(L, t)$ decays exponentially with rate Γ_p . This qualitative behavior is verified by a numerical evaluation of the probability density Eq. (88) in the regime $\sigma_p v_p \Delta t \sim 1$ —see, Fig. 3.

4.3.4 Resonant tunneling

The results of Secs. 4.3.2 and 4.3.3 rely on the assumption that the particles' momentum p is far from the resonances of the double barrier, Eq. (81). Here, we consider the probability distribution $P(L, t)$ for an initial state centered on a momentum p that is close to a resonance frequency k_0 .

Near the resonance $k = k_0$, the transmission amplitude is approximated by

$$T_k = \frac{|T_{0k_0}|^2 e^{2i\phi_k}}{1 - |R_{0k_0}|^2 e^{2i(k-k_0)(r+a+\phi'_{k_0})}}. \quad (102)$$

For $|T_{0k_0}| \ll 1$, the modulus square $|T_k|^2$ is of the order of $|T_{0k_0}|^4$ except for a narrow strip of momenta around k_0 where $|T_k|^2$ approaches unity. These momenta will dominate in the probability

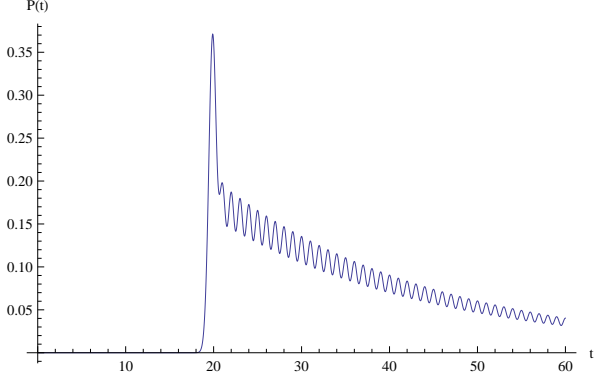


Figure 3: The probability density $P(L, t)$, Eq. (88), as a function of the time t , in the regime $\sigma_p v_p \Delta t \sim 1$. The probability exhibits a sharp peak at the time t_0 of first detection, followed by a sequence of shorter peaks that decay exponentially with rate Γ_p , given by Eq. (92).

distribution $P(L, t)$. Hence, we expand the phase factor $e^{2i(k-k_0)(r+a+\phi'_{k_0})}$, keeping only first order terms with respect to $k - k_0$. The transition amplitude becomes

$$T_k = \frac{e^{2i\phi_k}}{1 - i\frac{2v_{k_0}}{\Gamma_{k_0}}(k - k_0)}, \quad (103)$$

where Γ_{k_0} is the decay coefficient, Eq. (92), evaluated at the resonance frequency.

In this approximation, the squared amplitude

$$|T_k|^2 = \frac{1}{1 + \frac{2v_{k_0}^2}{\Gamma_{k_0}^2}(k - k_0)^2}. \quad (104)$$

is a Lorentzian of width $\Gamma_{k_0}/(2v_{k_0})$ centered around $k = k_0$.

We consider a initial wave-packet centered around momentum $k = p$, such that p lies within the peak of the Lorentzian Eq. (104). The probability density $P(L, t)$ is obtained from the squared modulus of the amplitude

$$\mathcal{A}(L, t) = \int_0^\infty \frac{dk}{2\pi} \sqrt{v_k} \frac{1}{1 - i\frac{2v_{k_0}}{\Gamma_{k_0}}(k - k_0)} \tilde{u}_0(k - p) e^{i(k(x_0+L) - E_k t + 2\phi_k)} \quad (105)$$

We assume that the momentum spread σ_p of $\tilde{\phi}_0$ is much smaller than p , so that we can enlarge the integration range to $(-\infty, \infty)$. We also expand the phase factor to first order in $q = k - p$, to obtain

$$\mathcal{A}(L, t) = \sqrt{v_p} e^{ip(x_0+L) - iE_p t + i\phi_p} \int_{-\infty}^\infty \frac{dq}{2\pi} \frac{1}{1 - i\frac{2v_{k_0}}{\Gamma_{k_0}}[q - (k_0 - p)]} \tilde{u}_0(k - p) e^{-iyv_p(t-t_0)}. \quad (106)$$

We calculate the integral in Eq. (106) analytically, by choosing a Lorentzian wave-packet

$$\tilde{u}_0(k) = \frac{1}{\sqrt{\sigma_p}} \frac{1}{1 + \frac{k^2}{\sigma_p^2}}. \quad (107)$$

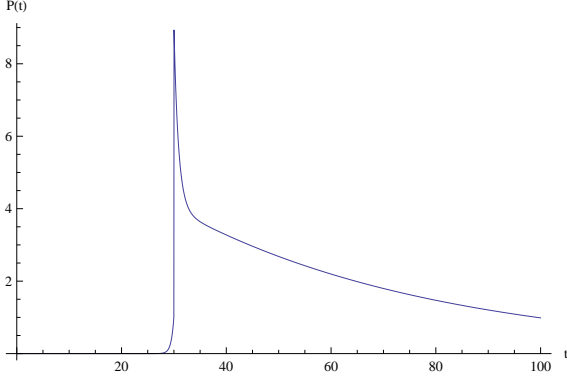


Figure 4: The probability density $P(L, t)$, Eq. (88), as a function of the time t , near resonance and for an initial Lorentzian wave-packet, Eq. (107). The probability exhibits a sharp peak at the time t_0 of first detection, and an exponential decay phase with rate Γ_{k_0} , given by Eq. (92).

Then, we obtain

$$\begin{aligned} \mathcal{A}(L, t) &= \frac{\frac{i\Gamma_{k_0}\sqrt{v_p}}{2\sqrt{\sigma_p v_{k_0}}} e^{ip(x_0+L)-iE_p t+i\phi_p}}{1 + \left(\frac{k_0-p}{\sigma_p} - i\frac{\Gamma_{k_0}}{2v_{k_0}\sigma_p}\right)^2} \\ &\times \begin{cases} \left[\frac{k_0-p}{\sigma_p} - i\left(\frac{\Gamma_{k_0}}{2v_{k_0}\sigma_p} - 1\right)\right] e^{-v_p\sigma_p|t-t_0|}, & t < t_0 \\ -2ie^{-iv_p(k_0-p)(t-t_0)-\frac{\Gamma_{k_0}}{2}|t-t_0|} + \left[\frac{k_0-p}{\sigma_p} - i\left(\frac{\Gamma_{k_0}}{2v_{k_0}\sigma_p} + 1\right)\right] e^{-v_p\sigma_p|t-t_0|}, & t > t_0 \end{cases} \quad (108) \end{aligned}$$

We note that the amplitude is suppressed for times $t < t_0$. The first detection signal appears around $t = t_0$. Again, this implies that the time delay due to the barrier is $2\phi'_p/v_p$.

The probability density $P(L, t) = |\mathcal{A}(L, t)|^2$ is computed straightforwardly. In the regime of exact resonance ($|k_0 - p| \ll \sigma_p$), and for $\Gamma_{k_0} \gg \sigma_p v_p$, it simplifies

$$P(L, t) \simeq \frac{\Gamma_{k_0}^2 v_p}{\sigma_p^2 v_{k_0}^2} \begin{cases} e^{-v_p\sigma_p|t-t_0|} & t < t_0 \\ 4e^{-\Gamma_{k_0}(t-t_0)} + e^{-2\sigma_p v_p(t-t_0)} + 4e^{-(\frac{\Gamma_{k_0}}{2} + \sigma_p v_p)(t-t_0)} \cos[v_p(k_0 - p)(t - t_0)], & t > t_0. \end{cases} \quad (109)$$

The instant t_0 of first detection is followed by a transient regime, and at times t such that $t - t_0 \gg (\sigma_p v_p)^{-1}$, the probability decays purely exponentially with rate Γ_{k_0} .

Hence, also in the case of resonance the double square barrier is characterized by two time-scales: the delay time $2\phi'_p/v_p$ and the decay rate Γ_{k_0} of the detection probability at later times.

4.3.5 Multiple resonances

For large inter-barrier distances, the difference between two successive resonance momenta is of the order of π/r . Hence, for wave packets with significant momentum spread $\sigma_p \sim \pi/r$, two or more resonances may contribute significantly to the detection probability. In this case, we express the transmission amplitude as

$$T_k = e^{2i\phi_k} \sum_n \frac{1}{1 - i\frac{2v_{k_n}}{\Gamma_{k_n}}(k - k_n)}, \quad (110)$$

where k_n correspond to the resonance frequencies.

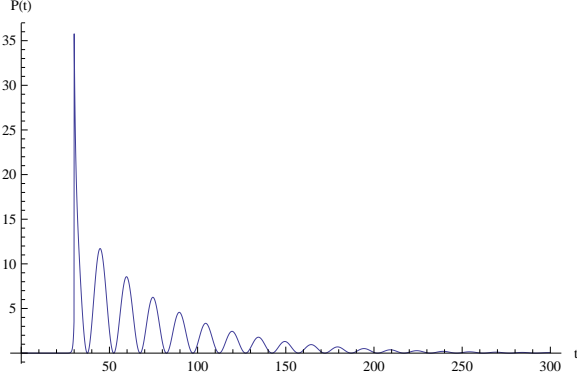


Figure 5: The probability density $P(L, t)$, Eq. (88), as a function of the time t , for an initial Lorentzian wave-packet, Eq. (107), that overlaps with two momentum resonances k_1 and k_2 . The probability exhibits a sharp peak at the time t_0 of first detection, and an exponential decay phase with rate $\Gamma_{k_1} \simeq \Gamma_{k_2}$, modulated by oscillations of frequency $\simeq v_p |k_2 - k_1|$.

Using the approximations employed in the derivation of Eq. (106), we obtain

$$\mathcal{A}(L, t) = \sqrt{v_p} e^{ip(x_0+L) - iE_p t + i\phi_p} \sum_n \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{1 - i \frac{2v_{kn}}{\Gamma_{kn}} [q - (k_n - p)]} \tilde{u}_0(k - p) e^{-iqv_p(t-t_0)}. \quad (111)$$

The amplitude $\mathcal{A}(L, t)$ is a sum of partial amplitudes $\mathcal{A}_n(t, L)$, each incorporating the contribution from a single resonance. For the Lorentzian initial state q , Eq. (107), each amplitude $\mathcal{A}_n(L, t)$ is of the form (108). The resulting probability distribution has the same structure as Eq. (108) around $t = t_0$, since t_0 depends only on the mean momentum of the wave-packet. However, at later times the behavior differs. To a first approximation, the decay constants at different channels are equal $\Gamma_{k_n} \simeq \Gamma_p$. Then, the dominant contribution to $P(L, t)$ at later times is

$$P(L, t) \sim \left| \sum_n e^{-iv_p(k_n - p)(t-t')} \right|^2 e^{-\Gamma_p(t-t_0)}. \quad (112)$$

Thus, at later times, the exponential decay of the probability density $P(L, t)$ is modulated by oscillations of frequencies $v_p(k_n - k_m) \sim v_p/r$.

4.3.6 Summary

We calculated the probability density $P(L, t)$ for the double-square barrier in different regimes and using different approximation schemes. In all regimes, $P(L, t)$ has the same qualitative behavior: it is characterized by a sharp peak at the time t_0 of first detection and (sometimes after a transient period) then decays exponentially to zero with a rate Γ_p . The probability distribution is determined by two parameters that are defined solely in terms of the double barrier's characteristics and the incoming particle momentum: the delay time $t_d(p)$ associated to first detection, and the decay rate Γ_p . The delay time $t_d(p)$ is twice the delay time for a single barrier. From the value of the delay time, one is led to a classical picture of tunneling, according to which the particles crosses the two barriers independently, each crossing taking time τ_p each, while propagating freely in the inter-barrier region.

The exponential decay in the probability density $P(L, t)$ at late times is due to the fraction of particles that cross the first barrier, and then they are trapped in the inter-barrier region. For these particles, the double barrier acts as a potential well, which they can only exit through tunneling. The exponential decay law at later times is of the same origin as the exponential decay of unstable

states through tunneling. Indeed, its derivation in Sec. 4.3.2 is rather similar to the classic studies of α -decay by Gamow [32] and Gurney and Condon [33]—see also, Ref. [20].

A key point that emerges from this analysis, is that the temporal aspects of tunneling cannot be captured by a single variable. The full construction of the probability distribution $P(L, t)$ is necessary. In particular, if one attempts to describe tunneling in terms of the time-delay obtained through a stationary phase approximation, as in Eq. (54) one is led to very different conclusions, namely, that there exists a regime, in which the inferred tunneling time is independent of the inter-barrier distance r , a property referred to as the "generalized Hartmann effect" [28, 29]. However, as shown in Ref. [34], a simple stationary phase approximation does not capture the properties of the Schrödinger evolved wave function in the double barrier system. A careful implementation of the stationary phase approximation in the amplitude $\mathcal{A}(L, t)$ of Eq. (85) leads to the results presented here.

5 Conclusions

In this work, we developed an approach to tunneling time based on the definition of probabilities for quantum temporal observables. The tunneling time is defined in terms of a probability distribution $P(L, t)$ associated to time-of-arrival experiments in a tunneling set-up. The probability distribution $P(L, t)$ contains all information about the temporal properties of the tunneling process.

We presented a methodology for constructing such probabilities, that applies to any quantum system. In particular, we applied this method to spinless relativistic particles described by a complex scalar quantum field. We derived a simple expression for the probability distribution $P(L, t)$, in which all information about the potential barrier is encoded in a complex coefficient A_k constructed from the transmission and reflection coefficients of the potential, T_k and R_k respectively. We found that the total transmission probability is proportional to $|A_k|^2$ rather than to $|T_k|^2$ and that this difference may be very pronounced for highly asymmetric potentials.

We evaluated the probability distribution $P(L, t)$ explicitly for piecewise constant potentials. The case of the double barrier potential is of particular significance, because it provides an explicit demonstration that a single parameter (like the tunneling time) does not suffice to describe all temporal aspects of the tunneling process.

In the present paper, we specialized to the standard case of a potential barrier that is defined in terms of a classical, background electric field. However, the applicability of the method is not restricted to this case. The probability distribution $P(L, t)$ is defined in terms of the two-point correlation function of any field theory with no restriction on the dynamics. Thus, the method presented here can be applied to a broader set of processes, for example, tunneling in open quantum systems (including the effect of dissipation and noise), or to incorporate the backreaction of the microscopic particles on the potential barrier.

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